Math 7580 project-

Instructor: Dr. Albert

Student: Jing-Yi Wu

*Douglas G. Bonett -- Confidence Intervals for Mean Absolute Deviations.*

**Introduction:**

Norma-theory tests and confidence intervals for variances are known to be hypersensitive to minor violations of the normality assumption. The mean absolute deviation an alternative way to give us informative measure of variability. Approximate confidence intervals for mean absolute deviations in one-group and two-group designs are derived are shown to have excellent small-sample properties under moderate non-normality.

**Methodology:**

Let  *(i = 1, 2, … ; j = 1, 2)* be continuous, independent and identically distributed random variables within group j with 0 < var() = < and E( ) = . The population median of is denoted as .

A consistent estimator of the mean absolute deviation is:

(1) , where is the sample mean

The estimator of is:

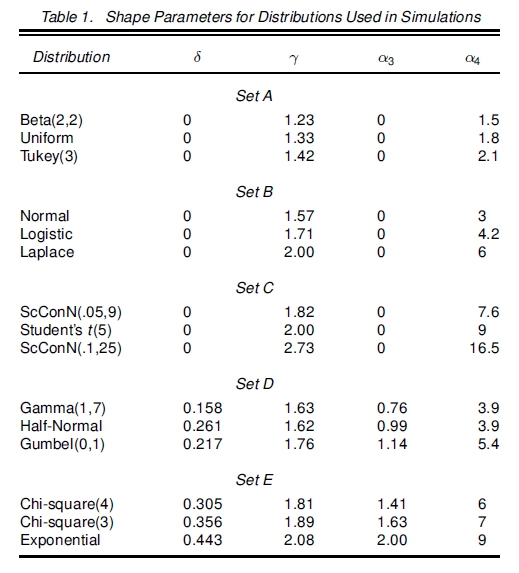
(2) , where = ()/ and

In a one-group design the following confidence interval is proposed

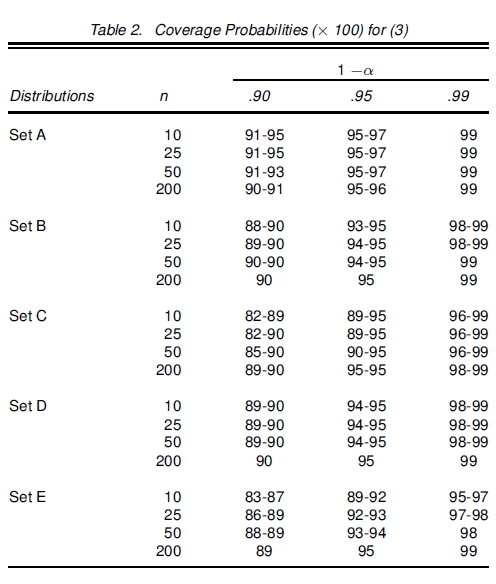
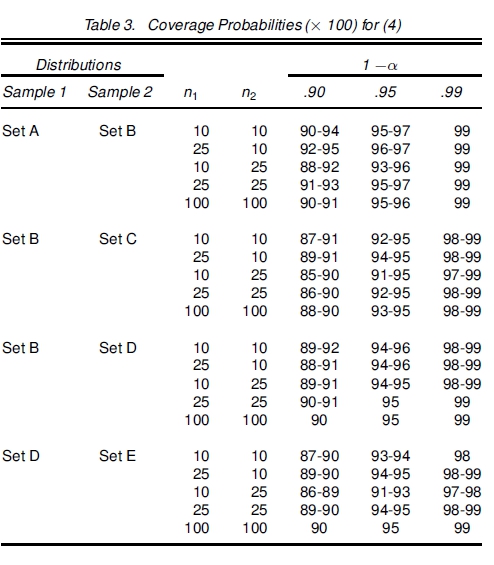
In a two-group design the following confidence interval is proposed

(4) , where

We want to find out the coverage probability for (3) and (4) when under different size(n) by utilizing Monte Carol method. Beside, we want to see if we can improve accuracy when we employ some techniques of variance reduction.

 (Table1. Is used in simulation.)

*\*\* Gamma(1=rate, 7=shape)*

Two tables above are author's results. Left table is one-group probability, and right table is two-sample group probability. I chose a distribution from in each set for one-group design and two-group design.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Set A | Set B | Set C | Set D | Set E |
| Uniform(0,1) | Normal(0,1) | Student's(5) | Gamma(7,1) | Chi-square(4) |

(Table 4)

**Monte Carol experiment:**

(I) One-group design:

This simulation study examined the small-sample performance of (3)

First, we need to know the true . By , we can calculate the true The Table1. helps us to get true , which is used in the simulation. Then, we sampled size n from a distribution to get the estimator of . For the estimator of, we used the formula (1) and (2) to get it. We finally got the value of estimators of and With the information from samples, we can construct the confidence interval. Repeat the procedure m times. We will have m sets of confidence intervals. The standard error is . is the rate that the number of confidence intervals really covering true . For example, the confidence interval covers 46,500 times under 50,000 iterations. The is 0.93, and the standard error is 0.0011.

The author used 50,000 Monte Carol random samples of a given sample size( n= 10, 25, 50) from a wide variety of distribution and 200. I also used Monte Carol study of 50,000 iterations. It’s time-consuming if I run all distributions in each set. I decided to run a distribution in each set.

I checked the value in Table1. From (1), we can estimator from the formula . I got a large samples from a distribution to get and , so we can get . Therefore, we can compare the and value in Table1. I noticed that the Gamma(1, 7) on the article is Gamma(1=rate, 7=shape) and Beta(2,2) is Beta (0.5=shape. 0.5=rate). I think the author should have labeled the shape and rate because it’s very confusing.

I wrote a function to get the confidence interval of (3). In the function, we will need to obtain n samples from a distribution for m times. From the samples, I could calculate for m times. Then, I wrote another function to see how many confidence intervals can cover the true . In the function, I calculated true and checked if those confidence intervals from previous function cover the . If it covers, it counted 1. The probability would be total times over m(=50,000), and the standard error would be . In set A, I chose Uniform(0, 1) distribution to represent set A. The variance is 1/12 and is 1.33. According the Formula, true is .2503. When is 0.1 and n=10, the 46,786 sets of confidence interval covers . Coverage probability is .93572, and S.E is .0011.

(II) Two-group design:

This simulation study examined the small-sample performance of (4)

We want to see if the constructed confidence interval can covers the ratio of from two distributions. Those two distributions represent two sets in Table 3. Two distribution are not required to be identical. By , we can calculate the ratio of and .The Table1. helps us to get true which is used in the simulation. If two distributions have similar shape such that =, (4) will generates confidence interval for .

We sampled size and from two distributions to get the .To estimate and , we used the formula (1) and (2) to get them. With the information from samples, we can construct the confidence interval to see if a confidence interval covers . Repeat the procedure for m times. We can generate m sets of confidence intervals.

The standard error is . is the coverage probability that confidence interval can cover . This simulation is to see how many constructed confidence intervals cover the .

The procedure is similar to one-group design. The can be calculated from the table. I wrote a function to find (4). In the function, we need to find out the estimators for , ), . I sampled n1 and n2 for distribution 1 and distribution 2 for m times. With these samples, we can construct (4). Then, I wrote another function to see how many confidence intervals can cover under 50,000 iterations, and the standard error would be . I chose Uniform(0,1) for Set A and N(0,1) for Set B. is 0.3136. When is 0.1 and n=10, there are 46,520 confidence intervals covering . Coverage probability is .9304, and S.E is .0011. I still use one distribution to represent a set as I concluded in Table 4.

**Result:**

Result for (I) one- group design *(Corresponded to Table 2)*

**Set A:** n alpha\_0.1 alpha\_0.05 alpha\_0.01

Coverage.Prob 10 93.57200 96.41800 98.80000

*Standard.Error 10 0.00110 0.00083 0.00049*

Coverage.Prob1 25 93.20400 96.76800 99.24800

*Standard.Error1 25 0.00113 0.00079 0.00039*

Coverage.Prob2 50 91.60400 95.92600 99.04800

*Standard.Error2 50 0.00124 0.00088 0.00043*

Coverage.Prob3 200 90.59600 95.38800 99.04800

*Standard.Error3 200 0.00131 0.00094 0.00043*

**Set B:** n alpha\_0.1 alpha\_0.05 alpha\_0.01

Coverage.Prob 10 91.25400 95.21400 98.51800

*Standard.Error 10 0.00126 0.00095 0.00054*

Coverage.Prob1 25 91.22600 95.40800 98.84400

*Standard.Error1 25 0.00127 0.00094 0.00048*

Coverage.Prob2 50 90.46600 95.14800 98.85600

*Standard.Error2 50 0.00131 0.00096 0.00048*

Coverage.Prob3 200 90.01400 95.05600 99.03000

*Standard.Error3 200 0.00134 0.00097 0.00044*

**Set C:** n alpha\_0.1 alpha\_0.05 alpha\_0.01

Coverage.Prob 10 87.92000 93.10000 97.75800

*Standard.Error 10 0.00146 0.00113 0.00066*

Coverage.Prob1 25 88.22000 93.49800 98.19600

*Standard.Error1 25 0.00144 0.00110 0.00060*

Coverage.Prob2 50 87.32800 92.84800 98.10200

*Standard.Error2 50 0.00149 0.00115 0.00061*

Coverage.Prob3 200 83.94600 90.62000 97.28000

*Standard.Error3 200 0.00164 0.00130 0.00073*

**Set D:** n alpha\_0.1 alpha\_0.05 alpha\_0.01

Coverage.Prob 10 89.92400 94.29000 98.16400

*Standard.Error 10 0.00135 0.00104 0.00060*

Coverage.Prob1 25 90.00600 94.62200 98.59400

*Standard.Error1 25 0.00134 0.00101 0.00053*

Coverage.Prob2 50 89.52000 94.45200 98.68400

*Standard.Error2 50 0.00137 0.00102 0.00051*

Coverage.Prob3 200 89.85200 94.87200 98.90200

*Standard.Error3 200 0.00135 0.00099 0.00047*

**Set E:** n alpha\_0.1 alpha\_0.05 alpha\_0.01

Coverage.Prob 10 86.82000 91.97800 97.12600

*Standard.Error 10 0.00151 0.00121 0.00075*

Coverage.Prob1 25 88.34000 93.37200 97.98800

*Standard.Error1 25 0.00144 0.00111 0.00063*

Coverage.Prob2 50 88.72800 93.82400 98.33000

*Standard.Error2 50 0.00141 0.00108 0.00057*

Coverage.Prob3 200 89.59600 94.72400 98.84800

*Standard.Error3 200 0.00137 0.00100 0.00048*

We can see the coverage probabilities will approach as the sample size increases. The S.E is small because the number of iterations is relatively big. When decreases, S.E also decreases. It makes sense that confidence covers more when decrease, since the range of lower and upper bound becomes wider.

I only ran a distribution to represent in each set. Most of coverage probabilities locate in the range in Table2. Few coverage probabilities are higher than author’s result.

Result for (II) two- group design *(Corresponded to Table 3)*

Set A and Set B:

n1 n2 alpha\_0.1 alpha\_0.5 alpha\_0.01

Coverage.Prob 10 10 93.04200 96.45800 99.16000

*Standard.Error 10 10 0.00114 0.00083 0.00041*

Coverage.Prob1 25 10 91.91400 95.74400 98.93400

*Standard.Error1 25 10 0.00122 0.00090 0.00046*

Coverage.Prob2 10 25 93.39800 96.74400 99.22400

*Standard.Error2 10 25 0.00111 0.00079 0.00039*

Coverage.Prob3 25 25 92.30800 96.27000 99.26800

Standard.Error3 25 25 0.00119 0.00085 0.00038

Coverage.Prob4 100 100 90.67200 95.33400 99.09600

*Standard.Error4 100 100 0.00130 0.00094 0.00042*

Set B and Set C:

n1 n2 alpha\_0.1 alpha\_0.5 alpha\_0.01

Coverage.Prob 10 10 90.52200 94.85000 98.61200

*Standard.Error 10 10 0.00131 0.00099 0.00052*

Coverage.Prob1 25 10 89.07400 93.97400 98.25800

*Standard.Error1 25 10 0.00140 0.00106 0.00059*

Coverage.Prob2 10 25 90.86800 95.14000 98.63200

*Standard.Error2 10 25 0.00129 0.00096 0.00052*

Coverage.Prob3 25 25 90.03600 94.73400 98.76800

*Standard.Error3 25 25 0.00134 0.00100 0.00049*

Coverage.Prob4 100 100 88.46600 93.71000 98.47400

*Standard.Error4 100 100 0.00143 0.00109 0.00055*

Set B and Set D:

n1 n2 alpha\_0.1 alpha\_0.5 alpha\_0.01

Coverage.Prob 10 10 91.24600 95.31600 98.79200

*Standard.Error 10 10 0.00126 0.00094 0.00049*

Coverage.Prob1 25 10 90.83400 95.00200 98.62800

*Standard.Error1 25 10 0.00129 0.00097 0.00052*

Coverage.Prob2 10 25 91.42800 95.55200 98.81200

*Standard.Error2 10 25 0.00125 0.00092 0.00048*

Coverage.Prob3 25 25 91.12200 95.47600 99.06400

*Standard.Error3 25 25 0.00127 0.00093 0.00043*

Coverage.Prob4 100 100 90.28800 95.27600 99.03400

*Standard.Error4 100 100 0.00132 0.00095 0.00044*

Set D and Set E:

n1 n2 alpha\_0.1 alpha\_0.05 alpha\_0.01

Coverage.Prob 10 10 89.38000 94.23800 98.41200

*Standard.Error 10 10 0.00138 0.00104 0.00056*

Coverage.Prob1 25 10 88.29600 93.30400 97.94200

*Standard.Error1 25 10 0.00144 0.00112 0.00063*

Coverage.Prob2 10 25 90.28600 94.68800 98.59200

*Standard.Error2 10 25 0.00132 0.00100 0.00053*

Coverage.Prob3 25 25 89.59200 94.59000 98.70200

*Standard.Error3 25 25 0.00137 0.00101 0.00051*

Coverage.Prob4 100 100 89.89800 94.69000 98.85800

*Standard.Error4 100 100 0.00135 0.00100 0.00048*

We can see the coverage probabilities will approach as the sample size increases in balanced and imbalanced designs. The standard errors are small because the number of iterations is relatively big.

**Variance Reduction:**

I tried three method for variance reduction for Monte Carol experiment. They are Antithetic Sampling, Importance Sampling and Control Variates. Unfortunately, these three method cannot work. I didn't apply variance reduction for two-group design because all three variance reduction techniques can't work for one-group design.

(I)Antithetic Sampling:

We find two identical distributed unbiased estimator. They are and , and they are negative correlated. Averaging two estimators will be superior to using either alone with double the sample size. The estimator will be. The variance of var(= , where is the correlation between and , and is the variance of either estimator using sample size n.

I tried uniform distribution from set A first. I sampled size n from uniform distribution for m/2 times, say For , they are 1-. Two sample paths are negative correlated under the setup. Since I have the 2 sets of samples, I can construct two confidence intervals. We want to see if two confidence interval can cover true . If the confidence interval covers true , we count 1. After we check (m/2) sets of confidence interval for, we sum all the counts for two confidence intervals. Averaging two numbers will be . If we divide by (m/2), it will be the coverage probability. The standard error of the coverage probability is the standard deviation of averaging count divided by square root of m. (). I changed a little based on one- group design. The point is to get two sample paths and see how many sets of confidence intervals which are constructed from two samples() cover true . Then, we collect the counts and average two numbers.

I set m =1000, since it takes so much time to run.

Naïve Monte Carol (Uniform (0,1) distribution Set A, m=1,000):

n alpha\_0.1 alpha\_0.05 alpha\_0.01

Coverage.Prob 10 94.00000 96.70000 99.30000

Standard.Error 10 0.00751 0.00565 0.00264

Coverage.Prob1 25 94.90000 96.80000 99.30000

Standard.Error1 25 0.00696 0.00557 0.00264

Coverage.Prob2 50 91.30000 95.50000 99.20000

Standard.Error2 50 0.00891 0.00656 0.00282

Coverage.Prob3 200 91.40000 95.50000 98.90000

Standard.Error3 200 0.00887 0.00656 0.00330

Antithetic Sampling (Unifrom(0,1) distribution Set A, m=1,000):

n alpha\_0.1 alpha\_0.05 alpha\_0.01

Coverage.Prob 10 94.60000 97.00000 99.20000

Standard.Error 10 0.00715 0.00540 0.00282

Coverage.Prob1 25 95.00000 98.20000 98.80000

Standard.Error1 25 0.00690 0.00421 0.00345

Coverage.Prob2 50 92.40000 95.80000 98.60000

Standard.Error2 50 0.00839 0.00635 0.00372

Coverage.Prob3 200 89.20000 94.40000 99.00000

Standard.Error3 200 0.00982 0.00728 0.00315

Naïve Monte Carol (Chi-square (4) distribution Set E, m=1,000):

n alpha\_0.1 alpha\_0.5 alpha\_0.01

Coverage.Prob 10 85.50000 91.90000 97.40000

Standard.Error 10 0.01113 0.00863 0.00503

Coverage.Prob1 25 87.90000 93.80000 98.40000

Standard.Error1 25 0.01031 0.00763 0.00397

Coverage.Prob2 50 87.50000 93.20000 98.70000

Standard.Error2 50 0.01046 0.00796 0.00358

Coverage.Prob3 200 89.80000 94.60000 99.20000

Standard.Error3 200 0.00957 0.00715 0.00282

Antithetic Sampling (Chi-square(4) distribution Set E, m=1,000):

n alpha\_0.1 alpha\_0.05 alpha\_0.01

Coverage.Prob 10 86.40000 93.00000 96.30000

Standard.Error 10 0.00822 0.00601 0.00438

Coverage.Prob1 25 89.30000 91.30000 95.90000

Standard.Error1 25 0.00708 0.00686 0.00468

Coverage.Prob2 50 88.40000 91.90000 97.20000

Standard.Error2 50 0.00732 0.00633 0.00377

Coverage.Prob3 200 89.40000 94.90000 99.10000

Standard.Error3 200 0.00741 0.00519 0.00210

The standard error doesn’t change a lot for Antithetic Sampling method in Set A. We can see some Standard error s are smaller than Monte Carol but some standard errors are even bigger. It doesn’t improve variance reduction. I think this it's related to the distribution. When I calculated the correlation between the counts from two confidence intervals. The correlation is 1. It means two confidence intervals, which is created from two sets of sample paths in Antithetic Sampling, are identical for uniform distribution. When I went back to test two confidence interval, I realized that two confidence intervals are the same because twos sets of samples are negative correlated and and from two sets sample are identical, too. This situation also happens in set B(Normal distribution) and Set C (t distribution). The correlation from two counts of confidence intervals is 1 for Normal distribution and are t distribution because they are symmetric. However, for set D(Gamma) and set E(Ch-square), the results changed. Since G(7,1) and Chi-square(4) are not symmetric, the and from two sets samples, which is created in Antithetic Sampling, are not identical. This two distributions can really improve the accuracy. From the Chiq-square table, when is 0.1 and n=10, the variance can be decreased by 88%. When is 0.01 and n=10, the antithetic sampling can reduce 31.8% of variance compared to Monte carol’s variance. I don’t think the result is desirable. I checked the correlation between two counts of two confidence intervals, and the correlation between two counts is 0.21. It’s not closed to -1. That’s the reason why Antithetic sampling cannot improve accuracy a lot for Chi-square distribution.

Since I found out that the Antithetic sampling method doesn't work well, I decided to try Importance sampling.

(II)Importance Sampling

The idea behind importance sampling is that certain values of the input [random variables](http://en.wikipedia.org/wiki/Random_variables) in a [simulation](http://en.wikipedia.org/wiki/Simulation) have more impact on the parameter being estimated than others. If these "important" values are emphasized by sampling more frequently, then the [estimator](http://en.wikipedia.org/wiki/Estimator) variance can be reduced.

In other words, the improved accuracy is achieved by causing the event of interest to occur more frequently than it would in the naïve Monte carol sampling framework.

Hence, the basic methodology in importance sampling is to choose a distribution which "encourages" the important values. The importance sampling method is based on the principal that the expectation of h(X) w.r.t its density can be written in the form

, where g is another density function, called envelope.

I tried chi-square(4) first. I used Chi-square(3.3) as envelope. (P\*100). Then chi-square(4)/chi-square(3.3) is the weight function( is weight function).I used the opposite way to calculate the coverage probability. If true is larger than upper bound of confidence. U is 1- weight; If true isn’t larger than upper bound of confidence, U =1. If true is smaller than lower bound of confidence. L is 1- weight; If true isn’t smaller than upper bound of confidence, L =1. U\*L is like the count is previous method. Finally, we sum (U\*L)/m. This is the coverage probability. The standard error is standard deviation of U\*L divided by square root of m ( )

I also set m=1,000 here.

Naïve Monte carol:

|  |  |  |  |
| --- | --- | --- | --- |
| Chi-square(4) |  |  |  |
| n=10 | 85.5 | 91.9 | 97.4 |
| n=10 | 0.01113 | 0.00863 | 0.00503 |

Importance Sampling:

|  |  |  |  |
| --- | --- | --- | --- |
| Chi-square(4) |  |  |  |
| n=10 | 86.8680 | 89.351 | 97.84 |
| n=10 | 0.0175 | 0.01925 | 0.0044 |

From the tables, we can see that the method doesn’t achieve variance reduction at all. This is two-sided problem. Here we want to find the coverage probability for two-sided confidence interval for . Also, for other distributions, it’s hard to find an appropriate distribution as an envelope to make it bounded.

(III) Control Variates:

The control variates strategy improves estimation of an unknown integral by relating to correlated estimator of an integral whose value is known.

Control variate estimator: ,  *is a parameter to be chosen by the user.*

=

I still used chi-square(4) as the first sample. The procedure is similar to Monte carol design. I construct the confidence interval to see how many times the confidence interval can cover over m iterations. But I need to record the variance(ss) from n samples for m times. After I get the counts and variances, I can create the regression model. With the model, I can get the . Then = counts + (ss-8), and 8 is the variance for Chi-square(4). Coverage probability is the sum of divided by m.

I set m=1,000 here.

Naïve Monte Carol:

|  |  |  |  |
| --- | --- | --- | --- |
| Chi-square(4) |  |  |  |
| n=10 | 85.5 | 91.9 | 97.4 |
| n=10 | 0.01113 | 0.00863 | 0.00503 |

Control variates:

|  |  |  |  |
| --- | --- | --- | --- |
| Chi-square(4) |  |  |  |
| n=10 | 87.42 | 92.79 | 97.386 |
| n=10 | 0.0105 | 0.00818 | 0.00503 |
| R-squared | 0.0005265 | 0.00010145 | 0.00170 |

We can see the Standard error does not change a lot. The S.Es from naïve Monte carol and Control variates are very close. The technique doesn’t improve accuracy. I noticed that the for regression model is extremely low when . I think that is the reason why it cannot reduce the variance. Theoretically, the control variates estimator is the fitted value on the regression model at the mean value of the predictor, and S.E of the control variates estimator is the S.E for the fitted value from the regression. In this case, if is small, the data are not close to fitted regression line. Lower tell us that the correlation between counts, which are the numbers when confidence covers , and variance of sample n is weak. This causes the failure at achieving variance reduction.

**Conclusions:**

I think the confidence intervals for and are not hard to compute. They are good alternatives for the classic confidence intervals for variances. We only need the Table2. to get the . The author uses the connection between and variance to

When I proceeded the techniques of variance reduction, I got very close estimates (The coverage probability) and Standard error. In the article, there are many distributions. That's why it's harder to find a good method to improve accuracy. Furthermore, this is two-sided confidence interval problem. In class, we usually consider one-sided problems. It definitely increases the difficulty of using most of variance reduction techniques for the Monte Carol in the article.

**Appendices:**Please see the attached file